

Penultimate Lecture (Claudia)

12 July 2016

Lurie proved a vast generalization of our main theorem

$$\text{Lie}_k \simeq \text{Moduli}_k \text{ (FMP)}$$

to many other settings.

His proof broke into two pieces

① formal categorical stuff

② specific facts about Lie algebras

(e.g. properties of C_{Lie}^*)

He axiomatizes part ① & then provides several analogs of ② (e.g. with associative algebras) to obtain

these generalizations

Today we'll describe this framework.

①

Example: of formal moduli problem

$$A \in \text{cdga}_k^{\leq 0}$$

Def An A -module M is projective of rank n if

① $\pi_0 M$ is a projective $\pi_0 A$ -module of rank n

② M is flat over A , i.e.,

$$\forall d, \text{Tor}_0(\pi_d A, \pi_0 M) \xrightarrow{\cong} \pi_d M.$$

Let $X_n: \text{cdga}_k^{\leq 0} \rightarrow \mathcal{S}$ be the functor

with

$$X_n(A) \subset \text{Mod}(A)$$

\uparrow

∞ -category of A -modules

objects are proj, rank n A -modules

higher simplices are "invertible morphisms"

This is clearly not a formal moduli problem but it's easy to extract one.

②

Let $k^{\oplus n}$ denote ~~the~~ n -dim vector space over k .

Define

$$X_n^1(A) := X_n(A) \times_{X_n(k)} \{k^{\oplus n}\}$$

Then

$$X_n^1 : \text{CAlg}_k^{\text{sm}} \rightarrow \mathcal{S}$$

is a formal moduli problem.

this is a
classifying space
of pairs
(M, α) with
 $\alpha: k \otimes_A M \xrightarrow{\cong} k^{\oplus n}$

It's useful to reformulate this problem:

Fact: if $A \in \text{CAlg}_k^{\text{sm}}$ & M is a connective A -module,
such that $M \otimes_A k \cong k^{\oplus n}$,
($\pi_i M = 0 \forall i \leq 0$)

then M is projective of rank n .

$$\Rightarrow X_n^1(A) = \text{Mod}_A^{\text{conn}} \times_{\text{Mod}_k^{\text{conn}}} \{k^{\oplus n}\}$$

This description actually makes sense even
for A associative but not commutative

by

$$X_n^1(A) = \text{LMod}_A^{\text{comm}} \times \{k^{\oplus n}\}$$

left commutative modules over associative dg alg A

$\text{LMod}_k^{\text{comm}}$

In fact, one can define an ∞ -category

$\text{Alg}_k^{\text{art}}$ of "Artinian dg algs"

(not necessarily commutative), and we

have

$$\tilde{X}_A^1 : \text{Alg}_k^{\text{art}} \longrightarrow \mathcal{S}$$

providing a "noncommutative formal moduli problem."

Q: How is the fact that we can extend from commutative to noncommutative setting reflected in the Lie algebra?

A: Let's begin by examining the tangent complex of X_n^1 .

$X_n^\wedge(k[\epsilon]/(\epsilon^2)) = \infty\text{-groupoid of first order defs of } k^{\oplus n}$
 i.e. rank n proj modules over dual numbers

$\simeq M/\epsilon M \cong k^{\oplus n}$

Fact: any basis of $k^{\oplus n}$ can be lifted to a basis of such a module M

\Rightarrow morphisms in this groupoid are of the form $\text{Id} + \mathcal{O}(\epsilon) = \text{Id} + \epsilon T$
 \downarrow
 identity on $k^{\oplus n}$

$T \in \text{End}(k^{\oplus n}) \cong \mathfrak{gl}_n$

In short,

$X_n^\wedge(k[\epsilon]/(\epsilon^2)) \cong B(G_n)$
 means simplicial set arising as classifying space of the discrete group G_n

$G_n = \{ \text{Id} + \epsilon \text{End}(k^n) \} \subset GL_n(k[\epsilon]/(\epsilon^2))$

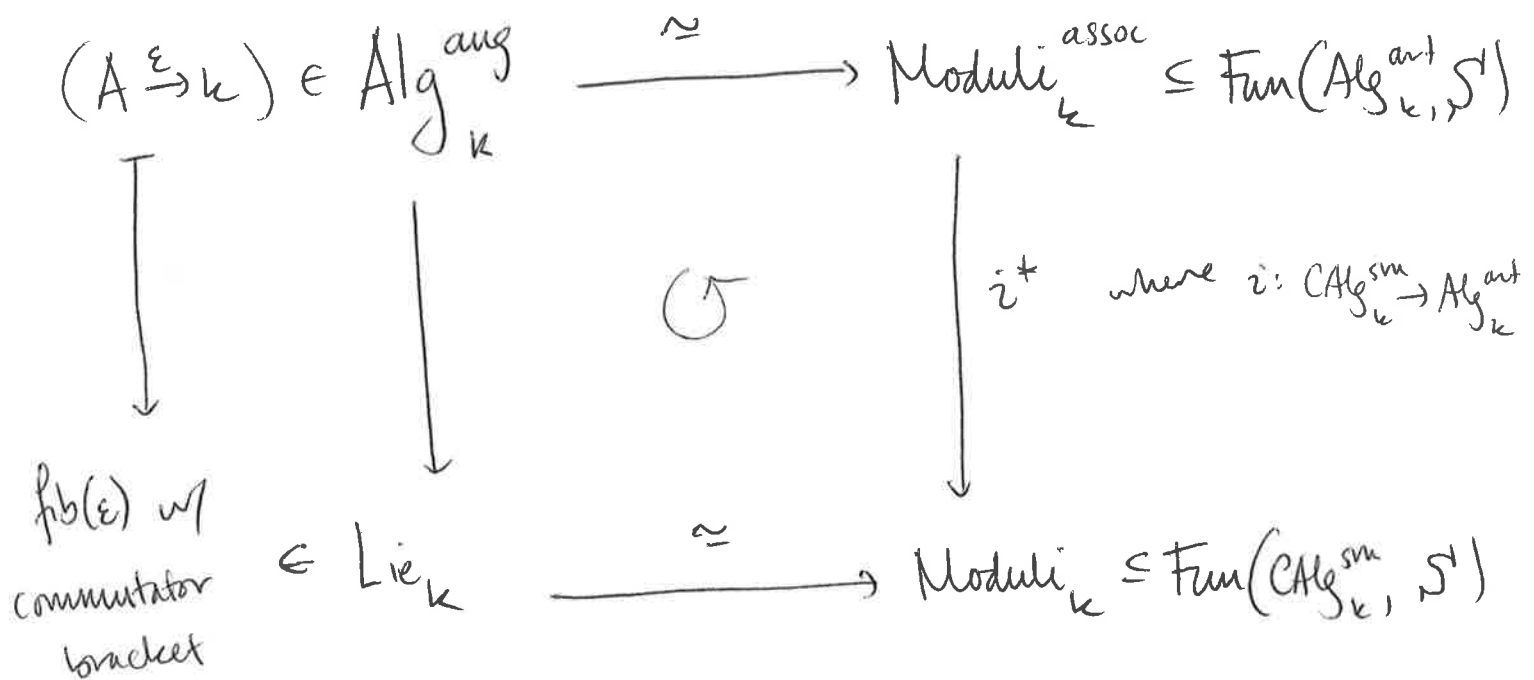
It should thus not be a surprise that the dg Lie algebra associated to X_n^\wedge is

$\text{End}(k^{\oplus n}) = \mathfrak{gl}_n.$

Why then can we extend to associative algs?

Because \mathfrak{gl}_n is the underlying Lie algebra of an associative algebra $\text{End}(k^{\oplus n})$. //

The general pattern is this (char $k=0$):



We want a general framework for understanding functors like the horizontal arrows above $\xrightarrow{\cong}$

We want something like:

\mathcal{A}, \mathcal{B} — ∞ -categories of algebraic objects
 \uparrow
 \mathcal{A}^{ant} "small algebras"

and then an equivalence

$$\mathcal{B} \xrightarrow{\cong} \text{Moduli}^{\mathcal{A}} \cong \text{Fun}(\mathcal{A}^{\text{aug}}, \mathcal{S}^1)$$

Lie = \mathcal{B} , $\mathcal{A} = \text{CAlg}$ etc

More specifically:

• \mathcal{A} — presentable ∞ -category

$$\mathcal{A} = \text{CAlg}_k^{\text{aug}}$$

• $\text{Sp}(\mathcal{A})$ — ∞ -category of spectra in \mathcal{A}

$$\text{Sp}(\mathcal{A}) \cong \text{Mod}_k$$

{ reduced & excisive functors

$$F: \mathcal{S}_+^{\text{fin}} \rightarrow \mathcal{A} \}$$

$$\text{Dual} \leftarrow k$$

$$\{k \oplus k[n]\}_{n \in \mathbb{N}}$$

w/ functor

$$\Omega^{\infty-n}: \text{Sp}(\mathcal{A}) \rightarrow \mathcal{A}$$

$$\Omega^{\infty-n}(\text{Dual}) \cong k \oplus k[n]$$

Def A deformation context is a pair $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$

where \mathcal{A} is a presentable ∞ -category and

$\{E_\alpha\}_{\alpha \in T}$ is a set of objects in $\text{Sp}(\mathcal{A})$.

$$(\text{CAlg}_k^{\text{aug}}, \text{Dual})$$

$T = *$

Def A map $\phi: A' \rightarrow A$ in \mathcal{A} is elementary

if $\exists \alpha \in T, n > 0$, such that there is a pullback square

$$\begin{array}{ccc} A' & \longrightarrow & * \\ \phi \downarrow & \text{P.B.} & \downarrow \\ A & \longrightarrow & \Omega^{\infty-n}(E_\alpha) \end{array}$$

terminal obj
in \mathcal{A}

eg.

$$\begin{array}{ccc} A' & \longrightarrow & k \\ \downarrow & \text{P.B.} & \downarrow \\ A & \longrightarrow & k \otimes k(n) \end{array}$$

Def A map $\phi: A' \rightarrow A$ in \mathcal{A} is small if it is a finite composition of elementary maps.

An object $A \in \mathcal{A}$ is small if $A \rightarrow *$ is small.

\mathcal{A}^{art} is the full subcategory of \mathcal{A} given by small ~~obj~~ objects.

e.g. $\text{CAlg}_k^{\text{sm}} \hookrightarrow \text{CAlg}_k^{\text{alg}}$

Example $\Omega^{\infty-n} E_\alpha$ is always small since

$$\begin{array}{ccc} \Omega^{\infty-n} E_\alpha & \longrightarrow & * \\ \downarrow & \text{P.B.} & \downarrow \\ * & \longrightarrow & \Omega^{\infty-n-1} E_\alpha \end{array}$$

Def A formal moduli problem for $(\mathcal{A}, \{E_\alpha\})$

is a functor $X: \mathcal{A}^{\text{art}} \rightarrow \mathcal{S}^1$ such that

① $X(*) \cong *$

② a pullback square σ in \mathcal{A}^{art}

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array} \quad \phi \leftarrow \text{small}$$

⑧

goes to a pullback square

$$\begin{array}{ccc} X(A') & \longrightarrow & X(B') \\ \downarrow & \text{PB} & \downarrow \\ X(A) & \longrightarrow & X(B) \end{array} \quad \text{in } \mathcal{S}$$

Let $\text{Moduli}^{\mathcal{A}} \subset \text{Fun}(\mathcal{A}^{\text{ant}}, \mathcal{S})$ be the full subcategory of formal moduli problems.

Ex $X_n^{\wedge} : \text{CAlg}^{\text{sm}} \longrightarrow \mathcal{S}$

$A \longmapsto$ rk n proj modules over A
s.t. fiber is $k^{\oplus n}$

Def Given $A \in \mathcal{A}$, let $\text{Spf}(A) \in \text{Fun}(\mathcal{A}^{\text{ant}}, \mathcal{S})$ be the functor

$$R \longmapsto \text{Maps}_A(A, R).$$

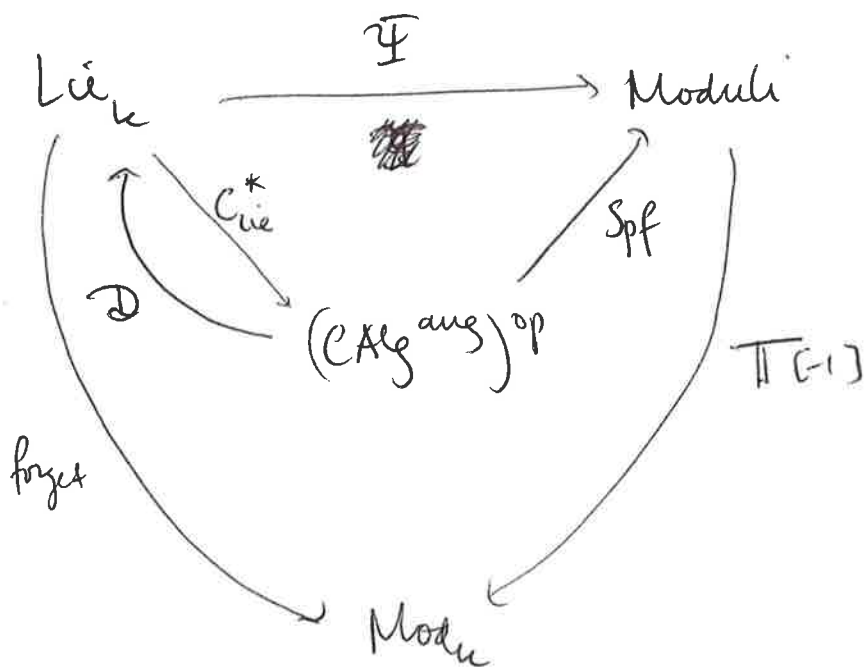
This is a "representable" functor & is in fact a formal moduli problem.

Hence we have a functor

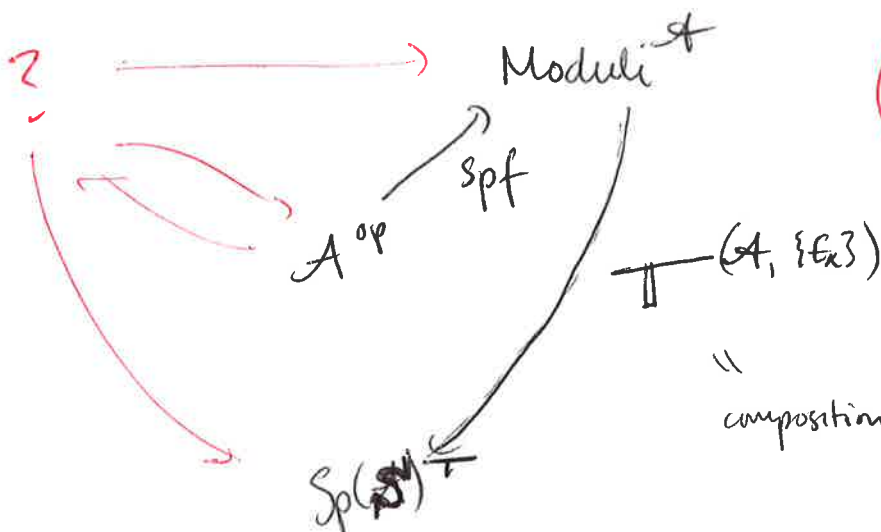
$$\text{Spf} : \mathcal{A}^{\text{op}} \longrightarrow \text{Moduli}^{\mathcal{A}}.$$

eg $\text{Spf} : (\text{CAlg}^{\text{aug}})^{\text{op}} \longrightarrow \text{Moduli}$

Recall our main diagram:



In general case, we need the analog of Lie_k , since we already have the RHS



main diagram

Here T^A is the tangent spectrum: for $Y \in Moduli^A$, the composites

$$S^1_{+} \xrightarrow{E_\alpha} A^{aut Y} \rightarrow S^1$$

give us the "tangents at $\alpha \in T$ "

Prop A map $u: X \rightarrow Y$ in Moduli^A is
an equivalence $\Leftrightarrow \mathbb{T}_X^A \xrightarrow{\mathbb{T}_X^A(u)} \mathbb{T}_Y^A$ is
an equivalence in $\text{Sp}(S^1)^T$

Here Lurie makes a clever maneuver:

he doesn't try to guess the general analog
of Lie_k but just introduces a definition.

The main work then boils down to
finding it! Here "classical" Koszul
duality helps.

Def A weak deformation theory for the
deformation context $(A, \{E_\alpha\}_{\alpha \in T})$ is a
functor $\mathcal{D}: \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$ such that

(D1) \mathcal{B} is a presentable ∞ -category (Lie_k)

(D2) \mathcal{D} admits a left adjoint $\mathcal{D}': \mathcal{B} \rightarrow \mathcal{A}^{\text{op}}$

(Civé) (11)

(D3) There is a full subcategory $\mathcal{B}_0 \subseteq \mathcal{B}$

of/ ① $\forall K \in \mathcal{B}_0$, the unit map

$$K \xrightarrow{\eta} \mathbb{D}\mathbb{D}'K$$

is an equivalence

② the initial object $\phi \in \mathcal{B}$ is in \mathcal{B}_0

③ for any $\alpha \in T$, $n \geq 1$, there is

$$K_{\alpha, n} \in \mathcal{B}_0 \text{ s.t. } \Omega^{\infty-n} E_{\alpha} \simeq \mathbb{D}'K_{\alpha, n}$$

↳ the basepoint of $\Omega^{\infty-n} E_{\alpha}$ determines

a map $v_{\alpha, n}: K_{\alpha, n} \simeq \mathbb{D}\mathbb{D}'K_{\alpha, n} \simeq \mathbb{D}(\Omega^{\infty-n} E_{\alpha}) \rightarrow \mathbb{D}(\ast) \simeq \phi$

$$\begin{array}{ccc} \textcircled{4} \text{ for every pushout } & K_{\alpha, n} & \xrightarrow{v_{\alpha, n}} & \phi & \text{ in } \mathcal{B} \\ & \downarrow & \text{P.O.} & \downarrow & \\ & K & \longrightarrow & K' & \end{array}$$

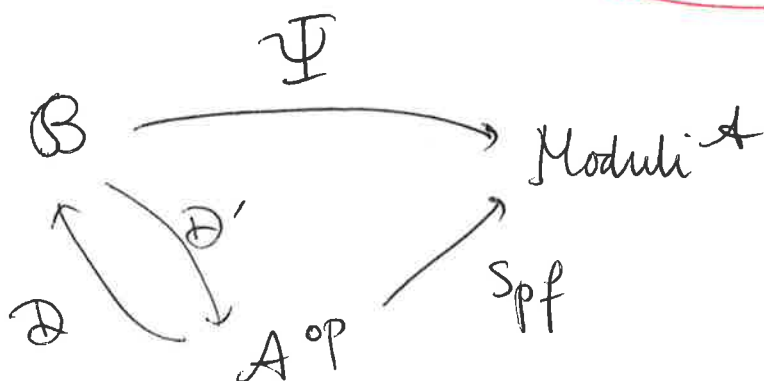
If $K \in \mathcal{B}_0$, then $K' \in \mathcal{B}_0$.

For Lie case, $K_{\alpha, n}$ were the $\mathfrak{g}^{(n)} = \text{Free}_{\text{Lie}}(k[-n-1])$,
which satisfied $C_{\text{Lie}}^+(\mathfrak{g}^{(n)}) \simeq k \oplus k[n]$

$$\mathbb{D}(K_{\alpha, n}) \quad \xrightarrow{\Omega^{\infty-n}} \quad \mathbb{D}(\text{Dual})$$

⑫

Given a weak deformation theory, we can try to complete main diagram



where $\Psi(\mathcal{B})/\mathcal{A} = \text{Maps}_{\mathcal{A}}(\Theta(\mathcal{A}), \mathcal{B})$

$$\Psi(\mathcal{B}) : \mathcal{A}^{\text{art}} \hookrightarrow \mathcal{A} \xrightarrow{\Theta^{\text{op}}} \mathcal{B}^{\text{op}} \xrightarrow{\text{Yoneda}} \mathbb{S}$$

ems of \mathcal{B}

Additional hypotheses provide sufficient (but not always necessary) conditions to mimic our proof that $\Psi : \text{Lie} \rightarrow \text{Moduli}$ was an equivalence

Def A deformation theory for $(\mathcal{A}, \{\epsilon_a\})$

is a weak deformation theory ~~with~~ $\mathcal{D} : \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$

such that

$$\textcircled{D4} \quad \forall \alpha \in T, \text{ let } e_\alpha : \begin{cases} \mathcal{B} \longrightarrow \text{Sp} \\ \mathcal{B} \longmapsto (\mathcal{S}_*^{\text{fin}} \xrightarrow{E_\alpha} \mathcal{A} \xrightarrow{\mathcal{D}} \mathcal{B}^{\text{op}} \xrightarrow{\text{Mod}_{\mathcal{A}}} \mathcal{S}^*) \end{cases}$$

Then we require e_α preserves small sifted colimits & reflects equivalences.

Theorem

for $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ a deformation context
and $\mathcal{D}: \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$ a deformation theory,
then the functor

$$\mathcal{Y}: \mathcal{B} \rightarrow \text{Moduli}^{\mathcal{A}}$$

is an equivalence of ∞ -categories.

Example $\mathcal{A} = \text{Alg}_k^{\text{aug}}$, $\text{Sp}(\mathcal{A}) \simeq \text{Mod}_k$

& let $\{k \otimes k[n]\}_{n \in \mathbb{N}} = \text{Dual}$ be the spectrum
(so $T = *$, $E_\alpha = \text{Dual}$)

$$\mathcal{B} = \text{Alg}_k^{\text{aug}} \quad \& \quad \mathcal{D}: \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$$

$(A \xrightarrow{\xi} k) \longmapsto \text{Hom}_A(k, k) \simeq (k \otimes k)^{\vee}$
is "usual" Koszul duality functor (14)