Penultimate Lecture (Claudia) 12 July 2016

Lurie proved a vast generalization of our main theorem

$$\mathrm{Lie}_k \simeq \mathrm{Moduli}_k \quad (\mathrm{FMP})$$

to many other settings.

His proof broke into two pieces

① formal categorical stuff

② specific facts about Lie algebras
(e.g. properties of C^*_{Lie})

He axiomatizes part ① & then provides several analogs of ② (e.g. with associative algebras) to obtain these generalizations

Today we'll describe this framework.

Example of formal moduli problem

$A \in \mathrm{cdga}_k^{\le 0}$

<u>Def</u> An A-module M is <u>projective of rank n</u> if

① $\pi_0 M$ is a projective $\pi_0 A$-module of rank n

② M is flat over A, i.e.,

~~$\forall$ [illegible]~~ $\forall d$, $\mathrm{Tor}_0(\pi_d A, \pi_0 M) \xrightarrow{\cong} \pi_d M$.

Let $X_n : \mathrm{cdga}_k^{\le 0} \longrightarrow S$ be the functor

with

$$X_n(A) \subset \mathrm{Mod}(A)$$

$\mathrm{Mod}(A)$: ∞-category of A-modules

$X_n(A)$: objects are proj, rank n A-modules; higher simplices are "invertible morphisms"

This is clearly not a formal moduli problem but it's easy to extract one.

Let $k^{\oplus n}$ denote ~~the~~ n-dim vector space over k.

Define

$$X_n^\wedge(A) := X_n(A) \times_{X_n(k)} \{k^{\oplus n}\}$$

this is a classifying space of pairs (M, α) with $\alpha: k \otimes_A M \xrightarrow{\simeq} k^{\oplus n}$

Then

$$X_n^\wedge : \mathrm{CAlg}_k^{sm} \longrightarrow \mathcal{S}$$

is a formal moduli problem.

It's useful to reformulate this problem:

Fact: if $A \in \mathrm{CAlg}_k^{sm}$ & M is a connective A-module, ($\pi_i M = 0 \ \forall i < 0$)

~~such that~~ $M \otimes_A k \cong k^{\oplus n}$,

then M is projective of rank n.

$$\Rightarrow X^\wedge(A) = \mathrm{Mod}_A^{conn} \times_{\mathrm{Mod}_k^{conn}} \{k^{\oplus n}\}$$

This description actually makes sense even for A <u>associative</u> but <u>not</u> commutative by

connective
left modules
over associative dg alg A

$$X_n^\wedge(A) = \mathrm{LMod}_A^{\mathrm{conn}} \times_{\mathrm{LMod}_k^{\mathrm{conn}}} \{k^{\oplus n}\}$$

In fact, one can define an ∞-category $\mathrm{Alg}_k^{\mathrm{art}}$ of "Artinian dg algs" (not necessarily commutative), and we have

$$\tilde{X}^\wedge_A : \mathrm{Alg}_k^{\mathrm{art}} \longrightarrow \mathcal{S}$$

providing a "noncommutative formal moduli problem."

__Q__: How is the fact that we can extend from commutative to noncommutative setting reflected in the Lie algebra?

__A__: Let's begin by examining the tangent complex of $X_n^\wedge$.

$\hat{X}_n(k[\varepsilon]/(\varepsilon^2))$ = ∞-groupoid of first order defs of $k^{\oplus n}$

i.e. rank n proj modules over dual numbers w/ $M/\varepsilon M \cong k^{\oplus n}$

__Fact__: any basis of $k^{\oplus n}$ can be lifted to a basis of such a module M

$\Rightarrow$ morphisms in this groupoid are of the form $\underbrace{Id}_{\text{identity on } k^{\oplus n}} + O(\varepsilon) = Id + \varepsilon T$

$T \in End(k^{\oplus n}) = gl_n$

In short,

$$\hat{X}_n(k[\varepsilon]/(\varepsilon^2)) \simeq \overbrace{B(G_n)}^{\text{means}}$$

means simplicial set arising as classifying space of the discrete group G_n

$$G_n = \{ Id + \varepsilon\, End(k^n) \} \subset GL_n(k[\varepsilon]/(\varepsilon^2))$$

It should thus not be a surprise that the dg Lie algebra associated to $\hat{X}_n$ is

$End(k^{\oplus n}) = gl_n$.

Why then can we extend to associative algs?

Because $\mathfrak{gl}_n$ is the underlying Lie algebra of an associative algebra $\mathrm{End}(k^{\oplus n})$. //

The general pattern is ~~this~~ this (char $k = 0$):

$$\begin{tikzcd}
(A \xrightarrow{\varepsilon} k) \arrow[d, mapsto] & \in \mathrm{Alg}^{aug}_k \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Moduli}^{assoc}_k \subseteq \mathrm{Fun}(\mathrm{Alg}^{art}_k, \mathcal{S}) \arrow[d, "i^*"] \\
\text{fib}(\varepsilon) \text{ w/ commutator bracket} & \in \mathrm{Lie}_k \arrow[r, "\simeq"] & \mathrm{Moduli}_k \subseteq \mathrm{Fun}(\mathrm{CAlg}^{sm}_k, \mathcal{S})
\end{tikzcd}$$

where $i : \mathrm{CAlg}^{sm}_k \to \mathrm{Alg}^{art}_k$

We want a general framework for understanding functors like the horizontal arrows above $\xrightarrow{\simeq}$

We want something like:

$\mathcal{A}, \mathcal{B}$ — ∞-categories of algebraic objects

$$\begin{matrix} \mathcal{A} \\ \uparrow \\ \mathcal{A}^{art} \end{matrix}$$

"small algebras"

and then an equivalence

$$\mathcal{B} \xrightarrow{\simeq} \mathrm{Moduli}^{\mathcal{A}} \subseteq \mathrm{Fun}(\mathcal{A}^{\mathrm{art}}, \mathcal{S})$$

$\mathrm{Lie} = \mathcal{B}$, $\mathcal{A} = \mathrm{CAlg}$ etc

<u>More specifically</u>:

- $\mathcal{A}$ — presentable ∞-category $\qquad \mathcal{A} = \mathrm{CAlg}_k^{\mathrm{aug}}$
- $\mathrm{Sp}(\mathcal{A})$ — ∞-category of spectra in $\mathcal{A}$ $\qquad \mathrm{Sp}(\mathcal{A}) \simeq \mathrm{Mod}_k$

$\quad \|$

$\{$reduced & excisive functors $F : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{A}\}$ $\qquad \mathrm{Dual} \longleftarrow k$

$\qquad \mathrm{Dual} = \{k \oplus k[d]\}_{d \in \mathbb{N}}$

w/ functor

$$\Omega^{\infty - n} : \mathrm{Sp}(\mathcal{A}) \to \mathcal{A} \qquad \Omega^{\infty - n}(\mathrm{Dual}) \simeq k \oplus k[n]$$

<u>Def</u> A <u>deformation context</u> is a pair $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ where $\mathcal{A}$ is a presentable ∞-category and $\{E_\alpha\}_{\alpha \in T}$ is a set of objects in $\mathrm{Sp}(\mathcal{A})$.

$$(\mathrm{CAlg}_k^{\mathrm{aug}}, \underbrace{\mathrm{Dual}}_{T = *})$$

Def A map $\phi : A' \to A$ in $\mathcal{A}$ is <u>elementary</u> if $\exists \alpha \in T$, $n > 0$, such that there is a pullback square

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\text{P.B.}"] & * \arrow[d] \\ A \arrow[r] & \Omega^{\infty-n}(E_\alpha) \end{tikzcd}$$

← terminal obj in $\mathcal{A}$

e.g.

$$\begin{tikzcd} A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & k \arrow[d] \\ A \arrow[r] & k \oplus k[n] \end{tikzcd}$$

Def A map $\phi : A' \to A$ in $\mathcal{A}$ is <u>small</u> if it is a finite composition of elementary maps.

An object $A \in \mathcal{A}$ is <u>small</u> if $A \to *$ is small.

$\mathcal{A}^{art}$ is the full subcategory of $\mathcal{A}$ given by small ~~objects~~ objects.

e.g. $\mathrm{CAlg}_k^{sm} \hookrightarrow \mathrm{CAlg}_k^{aug}$

<u>Example</u> $\Omega^{\infty-n} E_\alpha$ is always small since

$$\begin{tikzcd} \Omega^{\infty-n} E_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & * \arrow[d] \\ * \arrow[r] & \Omega^{\infty-n-1} E_\alpha \end{tikzcd}$$

Def A <u>formal moduli problem for $(\mathcal{A}, \{E_\alpha\})$</u> is a functor $X : \mathcal{A}^{art} \to \mathcal{S}$ such that

① $X(*) \simeq *$

② a pullback square σ in $\mathcal{A}^{art}$

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d, "\phi"] \\ A \arrow[r] & B \end{tikzcd}$$

ϕ ← small

,

goes to a pullback square
$$\begin{tikzcd} X(A') \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & X(B') \arrow[d] \\ X(A) \arrow[r] & X(B) \end{tikzcd}$$
in $\mathcal{S}$.

Let $\mathrm{Moduli}^{\mathcal{A}} \subset \mathrm{Fun}(\mathcal{A}^{\mathrm{art}}, \mathcal{S})$ be the full subcategory of formal moduli problems.

<u>Ex</u> $X_n^\wedge : \mathrm{CAlg}^{\mathrm{sm}} \longrightarrow \mathcal{S}$

$A \longmapsto$ rk n proj modules over A s.t. fiber is $k^{\oplus n}$

<u>Def</u> Given $A \in \mathcal{A}$, let $\mathrm{Spf}(A) \in \mathrm{Fun}(\mathcal{A}^{\mathrm{art}}, \mathcal{S})$ be the functor
$$R \longmapsto \mathrm{Maps}_{\mathcal{A}}(A, R).$$
This is a "representable" functor & is in fact a formal moduli problem.

Hence we have a functor
$$\mathrm{Spf} : \mathcal{A}^{op} \longrightarrow \mathrm{Moduli}^{\mathcal{A}}.$$

eg $\mathrm{Spf} : (\mathrm{CAlg}^{\mathrm{aug}})^{op} \longrightarrow \mathrm{Moduli}$

Recall our main diagram:

$$\begin{tikzcd}
\mathrm{Lie}_k \arrow[rr, "\Phi"] \arrow[dr, shift left, "C^*_{\mathrm{Lie}}"] \arrow[ddr, bend right, "\text{forget}"'] & & \mathrm{Moduli} \arrow[ddl, bend left, "T[-1]"] \\
& (\mathrm{CAlg}^{aug})^{op} \arrow[ul, bend left, "\mathfrak{D}"] \arrow[ur, "\mathrm{Spf}"'] & \\
& \mathrm{Mod}_k &
\end{tikzcd}$$

In general case, we need the analog of Lie_k, since we already have the RHS

$$\begin{tikzcd}
? \arrow[rr] \arrow[dr, bend left] \arrow[ddr, bend right] & & \mathrm{Moduli}^{\mathcal{A}} \arrow[ddl, bend left, "T^{(\mathcal{A}, \{E_\alpha\})}"] \\
& \mathcal{A}^{op} \arrow[ul, bend left] \arrow[ur, "\mathrm{Spf}"'] & \\
& \mathrm{Sp}(\mathcal{S})^T &
\end{tikzcd}$$

main diagram

$T^{(\mathcal{A}, \{E_\alpha\})}$ $=$ composition w/ the E_α

Here $T^{\mathcal{A}}$ is the tangent spectrum: for $Y \in \mathrm{Moduli}^{\mathcal{A}}$, the composites

$$S^{fin}_* \xrightarrow{E_\alpha} \mathcal{A}^{aug} \xrightarrow{Y} S$$

give us the "tangents at $\alpha \in T$"

<u>Prop</u> A map $u : X \longrightarrow Y$ in $\mathrm{Moduli}^{\mathcal{A}}$ is an equivalence $\Leftrightarrow$ $\amalg_X^{\mathcal{A}} \xrightarrow{\amalg^{\mathcal{A}}_{(u)}} \amalg_Y^{\mathcal{A}}$ is an equivalence in $Sp(S')^T$

Here Lurie makes a clever maneuver:
he doesn't try to guess the general analog of Lie_k but just introduces a definition.
The main work then boils down to finding it! Here "classical" Koszul duality helps.

<u>Def</u> A weak <u>deformation theory</u> for the deformation context $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ is a functor $\mathfrak{D} : \mathcal{A}^{op} \longrightarrow \mathcal{B}$ such that

(D1) $\mathcal{B}$ is a presentable ∞-category (Lie_k)

(D2) $\mathfrak{D}$ admits a left adjoint $\mathfrak{D}' : \mathcal{B} \to \mathcal{A}^{op}$ (C^*_{Lie})

(D3) There is a full subcategory $\mathcal{B}_0 \subseteq \mathcal{B}$

w/ (1) $\forall K \in \mathcal{B}_0$, the unit map

$$K \longrightarrow \mathcal{D}\mathcal{D}'K$$

is an equivalence

(2) the initial object $\phi \in \mathcal{B}$ is in $\mathcal{B}_0$

(3) for any $\alpha \in T$, $n \geq 1$, there is

$K_{n,\alpha} \in \mathcal{B}_0$ s.t. $\Omega^{\infty-n} E_\alpha \simeq \mathcal{D}' K_{\alpha,n}$

↳ the basepoint of $\Omega^{\infty-n} E_\alpha$ determines

a map $v_{\alpha,n} : K_{\alpha,n} \simeq \mathcal{D}\mathcal{D}'K_{\alpha,n} \simeq \mathcal{D}(\Omega^{\infty-n} E_\alpha) \to \mathcal{D}(*) \simeq \phi$

(4) for every pushout
$$\begin{tikzcd} K_{\alpha,n} \arrow[r, "v_{\alpha,n}"] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & \phi \arrow[d] \\ K \arrow[r] & K' \end{tikzcd}$$
in $\mathcal{B}$

if $K \in \mathcal{B}_0$, then $K' \in \mathcal{B}_0$.

For Lie case, $K_{\alpha,n}$ were the $\mathfrak{g}^{(n)} = \mathrm{Free}_{\mathrm{Lie}}(k[-n-1])$,

which satisfied $C^*_{\mathrm{Lie}}(\mathfrak{g}^{(n)}) \simeq k \oplus k[n]$

$\mathcal{D}'(K_{n,\alpha})$ $\qquad$ $\Omega^{\infty-n}$ (Dual)

Given a weak deformation theory, we can try to complete main diagram

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "\Psi"] \arrow[dr, bend left, "\mathfrak{D}'"] & & \mathrm{Moduli}^{\mathcal{A}} \\ & \mathcal{A}^{op} \arrow[ur, "\mathrm{Spf}"'] \arrow[ul, bend left, "\mathfrak{D}"] & \end{tikzcd}$$

where $\Psi(B)(A) = \mathrm{Maps}_{\mathcal{A}}(\mathfrak{D}(A), B)$

$$\Psi(B) : \mathcal{A}^{art} \hookrightarrow \mathcal{A} \xrightarrow{\mathfrak{D}^{op}} \mathcal{B}^{op} \xrightarrow[\text{Yoneda}]{\text{ems of } \mathcal{B}} \mathcal{S}$$

Additional hypotheses provide sufficient (but not always necessary) conditions to mimic our proof that $\Psi : \mathrm{Lie} \to \mathrm{Moduli}$ was an equivalence

__Def__ A <u>deformation theory</u> for $(\mathcal{A}, \{E_\alpha\})$ is a weak deformation theory ~~[illegible]~~ $\mathfrak{D} : \mathcal{A}^{op} \to \mathcal{B}$

such that

(D4) $\forall \alpha \in T$, let e_α : $\begin{cases} \mathcal{B} \longrightarrow Sp \\ B \longmapsto \left(\mathcal{S}_*^{fin} \xrightarrow{E_\alpha} \mathcal{A} \xrightarrow{\mathfrak{D}} \mathcal{B}^{op} \xrightarrow{\text{Yoneda}_B} \mathcal{S} \right) \end{cases}$

Then we require e_α preserves small sifted colimits & reflects equivalences.

Theorem

For $(\mathcal{A}, \{E_\alpha\}_{\alpha \in T})$ a deformation context and $\mathfrak{D} : \mathcal{A}^{op} \longrightarrow \mathcal{B}$ a deformation theory, then the functor

$$\Psi : \mathcal{B} \longrightarrow \text{Moduli}^{\mathcal{A}}$$

is an equivalence of ∞-categories.

Example $\mathcal{A} = Alg_k^{aug}$, $Sp(\mathcal{A}) \simeq Mod_k$

& let $\{k \oplus k[n]\}_{n \in \mathbb{N}} = \text{Dual}$ be the spectrum (so $T = *$, $E_\alpha = \text{Dual}$)

$\mathcal{B} = Alg_k^{aug}$ & $\mathfrak{D} : \mathcal{A}^{op} \longrightarrow \mathcal{B}$

$$(A \xrightarrow{\varepsilon} k) \longmapsto \text{Hom}_A(k, k) \simeq (k \otimes_A k)^\vee$$

is "usual" Koszul duality functor